Instability of the forced magnetohydrodynamics system at small Reynolds number

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Abstract

The aim of this paper is to give a result concerning the instability properties of the solutions of forced magnetohydrodynamics equations at small but finite Reynolds numbers. These solutions are highly oscillating spatially on the scale of the underlying flow but are growing on a larger scale depending on the magnetic Reynolds number. The existence of such solutions is called α-effect in the physical litterature. We prove instability results for a dense subset of initial velocity field of the flow at given Reynolds number.

Keywords: magnetohydrodynamics, stability, α-effect

1 Introduction

Dynamo theory first interested physicists almost one century ago, when Sir Joseph Larmor [14] wondered how a magnetic field could be generated by a rotating body like the Sun or the Earth. Since then, many results have come, both in mathematics and physics, to understand this kind of phenomena.

The physical model is a coupled system derived from the interaction of a conductive fluid, which follows Navier-Stokes equations, and an electromagnetic field, which follows Maxwell equations and Ohm's law. The interaction of the two physic quantities comes from the fact that a magnetic field has an effect on a moving conductive fluid, which creates in return a magnetic field. This kind of system is known as magnetohydrodynamics (MHD).

We have mainly two behaviors for the solutions of the MHD system: the magnetic energy can either collapse (exponentially), or stay/grow. In the latter case we say we have a dynamo, and it is the effect we observe in bodies like the Sun. However, the understanding of dynamo mechanisms is not simple. Indeed, the most "intuitive" way we imagine the conductive fluid in rotation and the magnetic field (that is a 2D rotating fluid in the kernel, and a magnetic field that looks like the one of a static magnet) is actually a

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well known anti-dynamo situation. Anti-dynamo theorems have been widely studied, and some summary of the knowledge in the domain has been made in [11].

The kind of situation we want to study there is that of the second kind, that is dynamo situation (and more precisely growth). Positive results began to emerge only later in the 50's, even though it was already well known that it is what occured in the Sun. In the 90's M.M. Vishik [23] proved that fast dynamos (i.e. exponential growth) require Lagrangian chaos in the fluid. Those results spread the idea that the velocity field has to be chaotic enough to expect a dynamo effect. Because of this "negative" result, different methods were tried to find instability results, linked to singularities: Y.B. Ponomarenko [17] studied concentration phenomena to get dynamo result in the case where Laplace force is neglected (i.e. imposed velocity field) and his results were extended by Gilbert [10] and successfully experimentaly checked [6]. More realistic case of the Ponomarenko dynamo was studied by D. Gérard-Varet and F. Rousset [9]. All those results give only asymptotic results.

In this paper we will focus on another singular dynamo mechanism, called *alpha effect*, first introduced by E.N. Parker [16], which is based on scale separation: we formally decompose the magnetic and the velocity fields into two parts: a fluctuating part, evolving on a turbulent scale, and a mean part, evolving on a larger scale. The mean value is broadly the spatial average over a sphere of radius much larger than the typical distance on which the field is turbulent, but much smaller thant the distance on which we want to observe the dynamo effect (in what follows, these distances will be 1 and $1/\varepsilon$, so scale separation will take place at small ε, which will be implicitly defined in term of R_m). This gives a decomposition of the velocity and magnetic fields under the form

$$
\begin{aligned}
u &= \tilde{u}(x, x/\varepsilon, t) + \bar{u}(x, t), \\
b &= \tilde{b}(x, x/\varepsilon, t) + \bar{b}(x, t),
\end{aligned}
\tag{1}
$$

where $\tilde{u}$ and $\tilde{b}$ have null mean value on the second variable. The idea behind *alpha effect* is that the averaged induction term $\overline{\nabla \wedge (\tilde{u} \wedge \tilde{b})}$ (we will specify those notations right below) that will appear in the MHD equations will give us a destabilizing effect on the magnetic field.

The *alpha effect* was first formally studied by G.O. Roberts [19, 18], in the framework of periodic flows. His results were discussed by H.K. Moffatt [15], S. Childress [2] and A.M. Soward [20] that considered a larger class of velocity fields and range of parameters. This effect has been since experimentaly confirmed by R. Stieglitz and U. Müller [21].

The aim of the present paper is to provide a rigorous justification of this mechanism, in a nonlinear framework. We will improve substantially former results of D. Gérard-Varet [7, 8].

Before stating more precisely our results, we will first recall the system we will work with:

The generation of magnetic field B by an electrically conducting fluid of velocity U and magnetic diffusivity ν_m follows the equations

$$
\begin{cases}
\partial_t B - \nabla \wedge (U \wedge B) - \nu_m \Delta B = 0, \\
\nabla \cdot B = 0.
\end{cases}
\tag{2}
$$

These equations are derived from the Maxwell's equation and Ohm's law (see second section of [3], or [11] for more detailed derivation). Since other relevant physics quantities that appear in the Maxwell's equations can be obtained from B, it is enough to work with only that.

Furthermore, the incompressible Navier-Stokes equations that drive the evolution of the conducting fluid under the force created by the magnetic field are

$$\begin{cases} \rho \partial_t U + \rho U \cdot \nabla U = -\nabla p + \eta \Delta U + \frac{1}{\mu_0}(\nabla \wedge B) \wedge B + f, \\ \nabla \cdot U = 0, \end{cases} \tag{3}$$

where p is the pressure field, f is an additional forcing term, ρ is the fluid mass density and η is the fluid shear viscosity (see [3] again, keeping in mind the incompressible condition). The additional forcing term will be chosen to generate some steady velocity field (and thus will be independent of time), see below.

We study there oscillating field U of amplitude V, oscillating on a space-scale L, and with magnetic diffusivity ν_m. After proper nondimensionalization of time, space, pressure and magnetic field, we then have the full incompressible MHD system:

$$\begin{cases} \partial_t B - \nabla \wedge (U \wedge B) - \frac{1}{R_m}\Delta B = 0, \\ \partial_t U + U \cdot \nabla U + \nabla p - \frac{1}{R_e}\Delta U = (\nabla \wedge B) \wedge B + f, \\ \nabla \cdot B = 0, \\ \nabla \cdot U = 0, \end{cases} \tag{4}$$

where $R_m = \frac{LV}{\nu_m}$ is the magnetic Reynolds number, and $R_e = \frac{\rho LV}{\eta}$ is the hydrodynamic Reynolds number.

Here we want to find a growing solution on a larger scale L/ε (where $\varepsilon > 0$), using the *alpha effect* introduced earlier. We will search this solution as a perturbation $\begin{pmatrix} u \\ b \end{pmatrix}$ of a solution $\begin{pmatrix} U_s \\ 0 \end{pmatrix}$ of this system (and the forcing term f is chosen accordingly so that U_s is a solution of the Navier-Stokes equation):

$$\begin{cases} U = u + U_s, \\ B = b + 0. \end{cases} \tag{5}$$

and the system in terms of b, u and U_s becomes

$$\begin{cases} \partial_t u + u \cdot \nabla U_s + U_s \cdot \nabla u - \frac{1}{R_e}\Delta u + \nabla p = (\nabla \wedge b) \wedge b - u \cdot \nabla u, \\ \partial_t b - \nabla \wedge (U_s \wedge b) - \nabla \wedge (u \wedge b) - \frac{1}{R_m}\Delta b = 0, \\ \nabla \cdot b = 0, \\ \nabla \cdot u = 0. \end{cases} \tag{6}$$

The space variables for this system is denoted θ, and corresponds to the fluctuation scale L, 1 after non-dimensionalization. In particular, we assume that

$$U_s = U_s(\theta) \in \mathcal{C}^\infty(\mathbb{T}^3) \tag{7}$$

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and has zero-mean.

Later on, we shall introduce a larger scale variable $x = \varepsilon\theta$ corresponding to the large scale L/ε of the growing field.

Our main result is an instability theorem in the Liapunov sense for system (6). To state our result, we introduce the set $\mathcal{H}_0^\infty(\mathbb{T}^3)$ of spatially 1-periodic functions of class $\mathcal{C}^\infty$ and null mean value with its usual distance:

$$d(U, U') = \sum_{k=0}^{\infty} \frac{\max\left(1, \|U - U'\|_{\mathcal{H}^k}\right)}{2^k}, \tag{8}$$

which makes it a Fréchet-space.

For $T = (T_1, T_2, T_3) \in (\mathbb{R}_+^*)^3$, we denote

$$\mathcal{H}_T^s = \mathcal{H}^s(\mathbb{R}/T_1\mathbb{Z} \times \mathbb{R}/T_2\mathbb{Z} \times \mathbb{R}/T_3\mathbb{Z}) \tag{9}$$

the inhomogeneous Sobolev space of order s on the torus

$$\mathbb{R}/T_1\mathbb{Z} \times \mathbb{R}/T_2\mathbb{Z} \times \mathbb{R}/T_3\mathbb{Z}. \tag{10}$$

We have the following theorem:

Theorem 1 *Let $s > 5/2$ be a real number. There exists a dense subset $\mathscr{P}$ of the unit ball of $\mathcal{H}_0^\infty(\mathbb{T}^3)$ and $R_m^0 > 0$ such that for all $U_s \in \mathscr{P}$ and $R_m < R_m^0$, for all $R_e > 0$, the solution $\begin{pmatrix} U_s \\ 0 \end{pmatrix}$ of the forced MHD system is nonlinearly unstable in the following sense:*

For some $T = (T_1, T_2, T_3) \in (\mathbb{N}^*)^3$ depending on U_s and R_m, and initial values

$$\begin{pmatrix} u_0 \\ b_0 \end{pmatrix} \in \mathcal{H}_T^s, \tag{11}$$

and $C_0 > 0$ such that for all $\delta > 0$, any solution $\begin{pmatrix} u_\delta \\ b_\delta \end{pmatrix}$ of (6) with initial value $\delta \begin{pmatrix} u_0 \\ b_0 \end{pmatrix}$ satisfies

$$\left\| \begin{pmatrix} u_\delta \\ b_\delta \end{pmatrix}(t_\delta) \right\|_{\mathcal{H}_T^s} \geqslant C_0 \tag{12}$$

for some time t_δ. Furthermore, we have the estimate on t_δ as $\delta \to 0$:

$$t_\delta \sim_{\delta \to 0} -\frac{\ln \delta}{C}, \tag{13}$$

where C is a positive constant.

T *will be defined in* (79)

As expected, this theorem expresses the instability of the forced MHD system, up to consider a larger periodic box $[0, T_1] \times [0, T_2] \times [0, T_3]$. Let us point out that $(T_1, T_2, T_3) \in (\mathbb{N}^*)^3$, so that U_s is also periodic on this larger scale. Note also that U_s is not necessarily a

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solution of the Navier-Stokes equation and that the forcing term f is chosen to compensate this.

In particular, for any

$$\begin{pmatrix} u_0 \\ b_0 \end{pmatrix} \in \mathcal{H}_T^s, \tag{14}$$

and any $\delta > 0$, system (6) has a unique solution

$$\begin{pmatrix} u_\delta \\ b_\delta \end{pmatrix} \in \mathcal{C}^0([0, T^*], H_T^s) \tag{15}$$

for some $T^* > 0$ with initial data $\delta(u_0, b_0)$. Moreover, for $\delta > 0$ small enough, it is well-known that one can take $T^* = +\infty$.

To the best of our knowledge, our theorem is the first full justification of the α-effect, in the nonlinear context.

In particular, it extends substantially the article [7] by David Gérard-Varet. Recast in our variables, this article considers the case of large magnetic diffusion, that is $R_m = \varepsilon \ll 1$. Briefly, it is shown that for all $m \in \mathbb{N}$, there are solutions of (6) (on a larger box depending on ε) that go from amplitude ε^m initially to an amplitude $\eta = \eta(m) > 0$ independent of ε (but depending on m).

Note that, as η depends on m, this result does not yield Liapunov instability at fixed small ε. Moreover, the study in [7] corresponds to vanishing magnetic Reynolds number. In our theorem, the threshold R_m^0 is of order 1, which is much more satisfactory from a physical point of view.

The outline of the paper is as follow:

The first and main part of the paper is the construction of an exact unstable eigenmode for the linearization of (6). It involves sharp spectral analysis arguments. Then, on the basis of this linear analysis, the proof of nonlinear instability is given in section 3.

2   Linear instability

In this section we will establish linear instability for the induction part of the MHD system. In other words, we will exhibit an unstable eigenmode of the linear operator

$$b \mapsto \nabla \wedge (U \wedge b) - \frac{1}{R_m} \Delta b \tag{16}$$

for a large class of U. This will be possible, through an accurate mathematical analysis of the *alpha effect* described in the introduction.

Let us write down the induction equation:

$$\partial_t b - \nabla \wedge (U \wedge b) - \frac{1}{R_m} \Delta b = 0, \\ \nabla \cdot b = 0, \tag{17}$$

where $U \in \mathcal{H}_0^\infty(\mathbb{T}^3)$ is independent of time.

In this section we prove the following linear instability result:

Theorem 2 *Let $s \geqslant 0$ be a real number. There exists a dense subset $\mathscr{P}$ of the unit ball of $\mathcal{H}_0^\infty(\mathbb{T}^3)$ and a positive real R_m^0 such that for all positive $R_m < R_m^0$, and all $U \in \mathscr{P}$, there exists $\xi \in (2\pi\mathbb{Q}^*)^3$ and a number $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, there exists a solution B of* (17) *of the form*

$$b(t,\theta) = e^{i\varepsilon\xi\cdot\theta}e^{\lambda^\varepsilon t}b^\varepsilon(\theta) \tag{18}$$

where $b^\varepsilon \in \mathcal{H}^s(\mathbb{T}^3)$, with exponential growth when $t \to +\infty$ (that is $\mathrm{Re}(\lambda^\varepsilon) > 0$).

2.1 Linearized equation

Let $U \in \mathcal{H}_0^\infty(\mathbb{T}^3)$ be a spatially periodic velocity field in $\mathbb{R}^3$ (of period 1).

We expect to record a growth of the magnetic field at large spatial scales $x = \varepsilon\theta$ (where ϵ will be defined in terms of R_m), and accordingly posit for the solution of (17) the ansatz

$$b(\theta,t) = e^{i\varepsilon\xi\cdot\theta+\varepsilon t\lambda^\varepsilon}B_0^\varepsilon(\theta), \tag{19}$$

for some $\xi \in \mathbb{R}^3$ and some λ^ε with $\mathrm{Re}\,\lambda^\varepsilon > 0$, where B_0^ε is 1-periodic in θ and has a power series expansion of the form:

$$B_0^\varepsilon(\theta) = \sum_{m\geqslant 0} \varepsilon^m(\bar{B}_m + \tilde{B}_m(\theta)), \qquad \int_{\mathbb{T}^3} \tilde{B}_m(\theta)\,\mathrm{d}\theta = 0. \tag{20}$$

We plug this ansatz in (17):

$$\begin{cases} \lambda^\varepsilon B_0^\varepsilon - \boldsymbol{i}\xi \wedge (U \wedge B_0^\varepsilon) - \frac{1}{\varepsilon}\nabla_\theta \wedge (U \wedge B_0^\varepsilon) + \frac{\varepsilon}{R_m}\xi^2 B_0^\varepsilon - \frac{1}{\varepsilon R_m}\Delta_\theta B_0^\varepsilon - \frac{2}{R_m}\boldsymbol{i}\sum_{i=1}^d \xi_i\partial_{\theta_i}B_0^\varepsilon = 0, \\ \boldsymbol{i}\xi \cdot B_0^\varepsilon + \frac{1}{\varepsilon}\nabla_\theta \cdot B_0^\varepsilon = 0, \end{cases} \tag{21}$$

and we solve according to powers of ε : we denote by $A^\varepsilon = A^0 + \varepsilon A^1 + \varepsilon^2 A^2$ the operator associated to the first equation of (21), that is:

$$A^0 = \nabla_\theta \wedge (U \wedge \cdot) + \frac{1}{R_m}\Delta_\theta$$

$$A^1 = \boldsymbol{i}\xi \wedge (U \wedge \cdot) + \frac{2}{R_m}\boldsymbol{i}\sum_{i=1}^d \xi_i\partial_{\theta_i} \tag{22}$$

$$A^2 = -\frac{1}{R_m}\xi^2\,\mathrm{Id},$$

where $A^\varepsilon \colon \mathcal{H}^{s+2}(\mathbb{T}^3) \to \mathcal{H}^s(\mathbb{T}^3)$, such that the first equation in (21) is

$$A^\varepsilon B_0^\varepsilon = \varepsilon\lambda^\varepsilon B_0^\varepsilon, \tag{23}$$

which is an eigenvalue problem. We will split this equation in powers of ε to obtain 3 equations.

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2.2 Study of the first term in the development

The first equation, corresponding to $\varepsilon = 0$, is

$$A^0 B_0^0 = 0, \tag{24}$$

that is $\frac{1}{R_m} \Delta_\theta B_0^0 + \nabla_\theta \wedge (U \wedge B_0^0) = 0$, which depends only on the independent variable θ.

We split B_0^0 between mean part and fluctuating part :

$$\frac{1}{R_m} \Delta_\theta \tilde{B}_0^0 + \nabla_\theta \wedge (U \wedge \tilde{B}_0^0) = -\nabla_\theta \wedge (U \wedge \bar{B}_0^0). \tag{25}$$

Since the Laplacian

$$\Delta_\theta \colon \mathcal{H}_0^{s+2}(\mathbb{T}^3) \subset \mathcal{H}_0^s(\mathbb{T}^3) \to \mathcal{H}_0^s(\mathbb{T}^3) \tag{26}$$

is an unbounded invertible operator, one can introduce

$$\Delta_\theta^{-1} \nabla_\theta \wedge (U \wedge \cdot) \colon \mathcal{H}_0^s(\mathbb{T}^3) \to \mathcal{H}_0^{s+1}(\mathbb{T}^3) \subset \mathcal{H}_0^s(\mathbb{T}^3) \tag{27}$$

which is a compact operator of domain $\mathcal{H}_0^s(\mathbb{T}^3)$.

We introduce the following operator, which is the restriction of A^0 to zero-mean functions:

$$L_{R_m} = \frac{1}{R_m} \Delta_\theta \cdot + \nabla_\theta \wedge (U \wedge \cdot) = \Delta_\theta \left(\frac{1}{R_m} \operatorname{Id} + \Delta_\theta^{-1} \nabla_\theta \wedge (U \wedge \cdot) \right). \tag{28}$$

By classical result on the spectrum of compact operators, L_{R_m} is invertible as an operator from $\mathcal{H}_0^{s+2}(\mathbb{T}^3)$ to $\mathcal{H}_0^s(\mathbb{T}^3)$ but for some countable set C_L of R_m which can be written as a sequence of numbers growing to $+\infty$. In particular it is invertible for R_m smaller than a given value R_m^0.

Since $-\nabla_\theta \wedge (U \wedge \bar{B}_0^0) \in \mathcal{H}_0^s(\mathbb{T}^3)$ for any $\bar{B}_0^0 \in \mathbb{R}^3$, the equation (25) admits exactly one solution for every $\bar{B}_0^0 \in \mathbb{R}^3$. Then for any R_m out of C_L, the equation

$$L_{R_m} \tilde{B}_0^0 = -\nabla_\theta \wedge (U \wedge \bar{B}_0^0) \tag{29}$$

is equivalent to

$$\tilde{B}_0^0 = \mathscr{L}(\theta) \bar{B}_0^0. \tag{30}$$

where $\mathscr{L}(\theta)$ is the operator from $\mathbb{R}^3$ to $\mathcal{H}_0^{s+2}(\mathbb{T}^3)$ defined by

$$\mathscr{L}(\theta) = -L_{R_m}^{-1} \nabla_\theta \wedge (U \wedge \cdot) \tag{31}$$

That means that the operator A^0 has a kernel of dimension 3, $A^0 \colon \mathcal{H}^{s+2}(\mathbb{T}^3) \subset \mathcal{H}^s(\mathbb{T}^3) \to \mathcal{H}^s(\mathbb{T}^3)$, that is $\ker A^0 = \{\bar{B} + \mathscr{L}(\theta)\bar{B}, \bar{B} \in \mathbb{R}^3\}$.

Using Lax-Milgram theorem and elliptic regularity, we know that the operator

$$\frac{1}{R_m} \Delta + \nabla_\theta \wedge (U \wedge \cdot) - \lambda \operatorname{Id} = A^0 - \lambda \operatorname{Id} \tag{32}$$

is invertible for large enough λ if seen as an operator from $\mathcal{H}_0^{s+2}$ to $\mathcal{H}_0^s$. Thus the resolvant of A^0 is compact in at least one λ, and all spectral projections are of finite dimension, that

is A^0 has eigenvalues of finite dimension, isolated and they are the only spectral values of A^0 (see [13] p.187).

Let σ be a smooth path surrounding 0 and no other eigenvalue of A^0. For ε sufficiently small, since A^ε varies continuously on ε, σ doesn't cross the spectrum of A^ε, and the operator

$$P^\varepsilon := \frac{1}{2i\pi} \int_\sigma (A^\varepsilon - \lambda \operatorname{Id})^{-1} \, d\lambda. \tag{33}$$

is the spectral projector associated to the eigenvalues of A^ε which tend to 0 as ε tends to 0. The eigenvalue 0 has multiplicity 3 in the spectrum of A^0. By continuity, this eigenvalue will vary – and maybe split – around 0 for positive ε, without crossing σ if ε is small enough. Note that the path depends on the choice of ξ.

Morover, still for ε small enough, P^ε also varies analytically on ε.

Furthermore, by definition, P^0 is the spectral projection of A^0 on its eigenvalue 0, and commutes with A^0 ([13] again, thm 6.17 p 178).

On $\operatorname{Im} P^0$, A^0 has then only 0 as an eigenvalue. Since $\operatorname{Im} P^0$ is of finite dimension, A^0 is nilpotent on this same space. Until the end of the subsection, we define $\mathcal{A}^0 = A^0{}_{|\operatorname{Im} P^0}$.

Let n be the integer such that

$$\ker \mathcal{A}^0 \subset \ker(\mathcal{A}^0)^2 \subset \cdots \subset \ker(\mathcal{A}^0)^n = \operatorname{Im} P^0, \tag{34}$$

We will now prove that $n = 1$, that is $\ker \mathcal{A}^0 = \ker(\mathcal{A}^0)^2$:
Let $B \in \ker(\mathcal{A}^0)^2$. Using the tilded and bar notation introduced in 2.1 to decompose a vector field in large-and-small-scale fields, we have the equivalences:

$$
\begin{aligned}
(\mathcal{A}^0)^2 B = 0 &\iff \mathcal{A}^0(\widetilde{\mathcal{A}^0 B}) + \mathcal{A}^0(\underbrace{\overline{\mathcal{A}^0 B}}_{=0}) = 0, \\
&\iff \widetilde{\mathcal{A}^0 B} = \bar{B}' + \mathscr{L}(\theta)\bar{B}' \text{ and } \overline{\mathcal{A}^0 B} = 0, \\
&\iff \widetilde{\mathcal{A}^0 B} = 0 \text{ and } \overline{\mathcal{A}^0 B} = 0, \\
&\iff \mathcal{A}^0 B = 0.
\end{aligned}
\tag{35}
$$

Thus $\ker \mathcal{A}^0 = \ker(\mathcal{A}^0)^2 = \operatorname{Im} P^0$.

2.3 Simplification of the equation

Back to (23), we look for solutions of that equation in $\operatorname{Im} P^\varepsilon$, that is in the sum of the eigenspaces corresponding to the spectral values of A^ε surrounded by σ:

$$P^\varepsilon A^\varepsilon P^\varepsilon B^\varepsilon = \varepsilon \lambda^\varepsilon B^\varepsilon. \tag{36}$$

Developping P^ε as the beginning of a series in ε, $P^\varepsilon = P^0 + \varepsilon P^1 + \varepsilon^2 P'^\varepsilon$, we have

$$P^\varepsilon A^\varepsilon P^\varepsilon = P^0 A^0 P^0 + \varepsilon P^1 A^0 P^0 + \varepsilon P^0 A^0 P^1 + \varepsilon P^0 A^1 P^0 + \varepsilon^2 A'^\varepsilon, \tag{37}$$

where both P'^ε and A'^ε depend analytically on ε.

Since P^0 and A^0 commute and $A^0 P^0 = 0$ (see results of previous subsection), the first three terms vanish, thus leaving us with the equation:

$$P^0 A^1 P^0 B^\varepsilon + \varepsilon A'^\varepsilon B^\varepsilon = \lambda^\varepsilon B^\varepsilon. \tag{38}$$

In the next subsections, we look for a solution of this equation for $\varepsilon = 0$. We shall then conclude using the continuity of eigenvalues as functions of ε.

2.4 Case $\varepsilon = 0$

The first order in ε of the equation (38) is equivalent to the system

$$\begin{cases} A^1 B = \lambda B \\ \tilde{B} = \mathscr{L}(\theta)\bar{B} \end{cases} \tag{39}$$

where $A^1 = \boldsymbol{i}\xi \wedge (U \wedge \cdot) - \frac{2\boldsymbol{i}}{R_m}\sum_{i=1}^{d}\xi_i \partial_{\theta_i}$ and $\mathscr{L}$ was introduced in (31). The mean part of the first equation is

$$\boldsymbol{i}\xi \wedge (\overline{U \wedge B}) = \lambda\bar{B}, \tag{40}$$

that is

$$\boldsymbol{i}\xi \wedge (\overline{U \wedge \mathscr{L}(\theta)\bar{B}}) + \underbrace{\boldsymbol{i}\xi \wedge (\overline{U \wedge \bar{B}})}_{=0} = \lambda\bar{B}, \tag{41}$$

$$\boldsymbol{i}\xi \wedge (\alpha\bar{B}) = \lambda\bar{B}, \tag{42}$$

where α is a constant matrix (that depends on U and R_m),

$$\alpha b = \int_{\theta \in \mathbb{T}^3} U(\theta) \wedge \mathscr{L}(\theta)b, \tag{43}$$

which we will study in the next subsection.

Thus we now have to find the eigenvalues of the operator $\boldsymbol{i}\xi \wedge (\alpha\cdot)$, that is to find the eigenvalues of the matrix $A^\xi \alpha$, where

$$A^\xi = \begin{pmatrix} 0 & -\boldsymbol{i}\xi_3 & \boldsymbol{i}\xi_2 \\ \boldsymbol{i}\xi_3 & 0 & -\boldsymbol{i}\xi_1 \\ -\boldsymbol{i}\xi_2 & \boldsymbol{i}\xi_1 & 0 \end{pmatrix}. \tag{44}$$

2.5 Study of the matrix α

The α matrix is defined for any $b \in \mathbb{R}^3$ by

$$\alpha b = \int_{\theta \in \mathbb{T}^3} U(\theta) \wedge \mathscr{L}(\theta)b = -\int_{\theta \in \mathbb{T}^3} U(\theta) \wedge \left(L_{R_m}^{-1}\nabla_\theta \wedge (U(\theta) \wedge b)\right). \tag{45}$$

We shall write this matrix as the sum of its skew symmetric part α^A and symmetric part α^S (which are both real).

Now we compute the eigenvalues of those matrices: since α^A is antisymmetric, his eigenvalues are $0, \pm \boldsymbol{i} \|V_U\|$ where V_U is a vector of $\mathbb{R}^3$ such that $\forall b \in \mathbb{R}^3, \alpha^A b = V_U \wedge b$.

For α^S, we will apply a slight variation to U so that the matrix changes to one of distinct non-zero eigenvalues (this doesn't change what we said about the antisymmetric part).

2.5.1 Symmetry of α

We have in the expression of α:

$$-L_{R_m}^{-1} \nabla_\theta \wedge (U \wedge b) = \left(\frac{1}{R_m} \operatorname{Id} - \mathscr{A} \right)^{-1} \left(-\Delta_\theta^{-1} \nabla_\theta \wedge (U \wedge b) \right) = \left(\frac{1}{R_m} \operatorname{Id} - \mathscr{A} \right)^{-1} \mathscr{A} b, \quad (46)$$

where

$$\mathscr{A} = -\Delta_\theta^{-1} \nabla_\theta \wedge (U \wedge \cdot) \colon \mathcal{H}_0^{s+2} \to \mathcal{H}_0^{s+3} \subset \mathcal{H}_0^{s+2}. \quad (47)$$

For small enough R_m (such that $1/R_m$ is bigger than the highest eigenvalue of the compact operator $\mathscr{A}$, say $R_m < 2R_m^0$ where $R_m^0 > 0$ – we will justify this factor 2 as a precaution at the end of the subsection), we can express the resolvent as a series:

$$-L_{R_m}^{-1} \nabla_\theta \wedge (U \wedge b) = \sum_{n=0}^{+\infty} R_m^{n+1} \mathscr{A}^{n+1} b = \sum_{n=1}^{+\infty} R_m^n \mathscr{A}^n b. \quad (48)$$

Furthermore, (47) gives in Fourier series, for $k \neq 0$ and $f \in \mathcal{H}^s(\mathbb{T}^3)$:

$$\widehat{\mathscr{A}f}(k) = -\frac{1}{2\pi} |k|^{-2} (-\boldsymbol{i}k) \wedge \left(\sum_{k'} \hat{U}(k') \wedge \hat{f}(k - k') \right). \quad (49)$$

Note that, by definition of Δ_θ^{-1}, $\widehat{Af}(0) = 0$.

Thus

$$\alpha b = \int_{\theta \in \mathbb{T}^3} U(\theta) \wedge \sum_{n=1}^{+\infty} (R_m \mathscr{A})^n b = \sum_{n=1}^{+\infty} R_m^n \int_{\theta \in \mathbb{T}^3} U(\theta) \wedge \mathscr{A}^n b, \quad (50)$$

and we recognize in the last part of the equation:

$$\int_{\theta \in \mathbb{T}^3} U(\theta) \wedge \mathscr{A}^n b = \widehat{U \wedge \mathscr{A}^n b}(0). \quad (51)$$

Recursively, we can compute $\widehat{U \wedge \mathscr{A}^n b}(0)$ and

$$\alpha b = \sum_{n=1}^{\infty} R_m^n \sum_{\sum_{j=1}^{n+1} k_j = 0} \hat{U}(k_{n+1}) \wedge \left(\frac{\boldsymbol{i}m_n}{2\pi |m_n|^2} \wedge \left(\hat{U}(k_n) \wedge \cdots \left(\frac{\boldsymbol{i}m_1}{2\pi |m_1|^2} \wedge \left(\hat{U}(k_1) \wedge b \right) \right) \cdots \right) \right), \quad (52)$$

where the second sum is actually restricted to tuples where all partial sums $m_i = \sum_{j=1}^{i} k_j$ are non-zero (otherwise there is a term of the form $\widehat{\mathscr{A}^k b}(0)$ which is zero by definition of $\mathscr{A}$).

This last equality is:

$$\alpha b = \sum_{n=1}^{\infty} R_m^n \alpha^{(n+1)} b, \tag{53}$$

where

$$\alpha^{(n)} = \sum_{\sum_{j=1}^{n} k_j = 0} \hat{U}(k_n) \wedge \left(\frac{\boldsymbol{i} m_{n-1}}{2\pi |m_{n-1}|^2} \wedge \left(\hat{U}(k_{n-1}) \wedge \cdots \left(\frac{\boldsymbol{i} m_1}{2\pi |m_1|^2} \wedge \left(\hat{U}(k_1) \wedge b \right) \right) \cdots \right) \right), \tag{54}$$

and $\alpha^{(n)}$ have the same symmetry as n (by recurrence, or see [1]).

2.5.2   Eigenvalues of α

The first symmetric matrix in power series is $\alpha^{(2)}$:
Let $U \in \mathcal{H}_0^\infty(\mathbb{T}^3)$, and $\eta > 0$. We want $\tilde{U}$ such that $\alpha^{(2)}(\tilde{U})$ is of distinct non-zero eigenvalues and $d(U, \tilde{U}) < \eta$, where d is the usual distance induced by semi-norms on $\mathscr{H}_0^\infty(\mathbb{T}^3)$: we define now

$$U^j(\theta) = \sum_{|k| \leqslant j} \hat{U}(k) e^{2ik\cdot\theta\pi}. \tag{55}$$

For large enough j, we have $d(U^j, U) \leqslant \eta/2$. Denote then $\tilde{U} = U^j + \sum_{i=1}^{3} \delta_i V^i$ where $\delta_i > 0$ is determined later, and

$$V^1 = \begin{pmatrix} \sin(2\pi(j+1)\theta_3) + \cos(2\pi(j+1)\theta_2) \\ \cos(2\pi(j+1)\theta_3) \\ \sin(2\pi(j+1)\theta_2) \end{pmatrix}, V^2 = \begin{pmatrix} \sin(2\pi(j+2)\theta_3) \\ \sin(2\pi(j+2)\theta_1) + \cos(2\pi(j+2)\theta_3) \\ \cos(2\pi(j+2)\theta_1) \end{pmatrix},$$
$$V^3 = \begin{pmatrix} \cos(2\pi(j+3)\theta_2) \\ \sin(2\pi(j+3)\theta_1) \\ \sin(2\pi(j+3)\theta_2) + \cos(2\pi(j+3)\theta_1) \end{pmatrix}. \tag{56}$$

Then:

$$\hat{V}^1(k)_y = \int e^{-2ik\cdot\theta\pi} V_y^1(\theta) \, d\theta = \begin{cases} 0 \text{ if } k_1 \neq 0 \text{ or } k_2 \neq 0 \\ \int_0^1 e^{-2i\pi k_3 \theta_3} \cos(2\pi(j+1)\theta_3) \, d\theta_3 \text{ otherwise} \end{cases}$$
$$= \begin{cases} 0 \text{ if } (k_3 \neq j+1 \text{ and } k_3 \neq -j-1) \text{ or } k_1 \neq 0 \text{ or } k_2 \neq 0 \\ \frac{1}{2} \text{ otherwise} \end{cases} \tag{57}$$

Thus

$$\hat{V}^1(k) = \begin{pmatrix} \frac{1}{2}\delta_{k_1=0}\delta_{k_3=0}(\delta_{k_2=j+1} + \delta_{k_2=-j-1}) + \frac{i}{2}\delta_{k_1=0}\delta_{k_2=0}(\delta_{k_3=-j-1} - \delta_{k_3=j+1}) \\ \frac{1}{2}\delta_{k_1=0}\delta_{k_2=0}(\delta_{k_3=j+1} + \delta_{k_3=-j-1}) \\ \frac{i}{2}\delta_{k_1=0}\delta_{k_3=0}(\delta_{k_2=-j-1} - \delta_{k_2=j+1}) \end{pmatrix}, \tag{58}$$

and similarly for the other ones:

$$\hat{V}^2(k) = \begin{pmatrix} \frac{i}{2}\delta_{k_1=0}\delta_{k_2=0}(\delta_{k_3=-j-2} - \delta_{k_3=j+2}) \\ \frac{1}{2}\delta_{k_1=0}\delta_{k_2=0}(\delta_{k_3=j+2} + \delta_{k_3=-j-2}) + \frac{i}{2}\delta_{k_2=0}\delta_{k_3=0}(\delta_{k_1=-j-2} - \delta_{k_1=j+2}) \\ \frac{1}{2}\delta_{k_2=0}\delta_{k_3=0}(\delta_{k_1=j+2} + \delta_{k_1=-j-2}) \end{pmatrix}, \quad (59)$$

$$\hat{V}^3(k) = \begin{pmatrix} \frac{1}{2}\delta_{k_1=0}\delta_{k_3=0}(\delta_{k_2=j+3} + \delta_{k_2=-j-3}) \\ \frac{i}{2}\delta_{k_2=0}\delta_{k_3=0}(\delta_{k_1=-j-3} - \delta_{k_1=j+3}) \\ \frac{1}{2}\delta_{k_2=0}\delta_{k_3=0}(\delta_{k_1=j+3} + \delta_{k_1=-j-3}) + \frac{i}{2}\delta_{k_1=0}\delta_{k_3=0}(\delta_{k_2=-j-3} - \delta_{k_2=j+3}) \end{pmatrix}. \quad (60)$$

For $V^i(k)$ to be non-zero, we need that $k_i = 0$ and $k_l \in \{0, j+i, -j-i\}$ when $l \neq i$.

Then $\hat{V}^1(-k) \wedge \left(i\frac{k}{|k|^2} \wedge \left(\hat{V}^2(k) \wedge B \right) \right) = 0$ as soon as $k_1 \neq 0$ or $k_2 \neq 0$. If both are null, then $\hat{V}^1(k) = 0$ but if $k_3 \in \{0, j+1, -j-1\}$, and $\hat{V}^2(k) = 0$ but if $k_3 \in \{0, j+2, -j-2\}$. Thus the product is always zero.

Only remains now non-crossed terms:

$$\hat{V}^1(-k) \wedge \left(i\frac{k}{|k|^2} \wedge \left(\hat{V}^1(k) \wedge B \right) \right) = \hat{V}^1(-k) \wedge \hat{V}^1(k).\left(i\frac{k}{|k|^2} \cdot B \right). \quad (61)$$

For it to be non-zero, we need that $k_1 = 0$. If furthermore $k_2 \neq 0$ and $k_3 \neq 0$, then $\hat{V}^1(k) = 0$.

If $k_3 = 0$ and $k_2 \neq 0$	If $k_3 \neq 0$ and $k_2 = 0$
$\hat{V}^1(k) = \begin{pmatrix} \frac{1}{2}(\delta_{k_2=j+1} + \delta_{k_2=-j-1}) \\ 0 \\ \frac{i}{2}(\delta_{k_2=-j-1} - \delta_{k_2=j+1}) \end{pmatrix}$	$\hat{V}^1(k) = \begin{pmatrix} \frac{i}{2}(\delta_{k_3=-j-1} - \delta_{k_3=j+1}) \\ \frac{1}{2}(\delta_{k_3=j+1} + \delta_{k_3=-j-1}) \\ 0 \end{pmatrix}$
Thus for $k_2 = j + 1$,	Thus for $k_3 = j + 1$,
$\hat{V}^1(-k) \wedge \hat{V}^1(k) = \frac{1}{2}\begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}$	$\hat{V}^1(-k) \wedge \hat{V}^1(k) = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix}$

and thus

$$\sum_{k \neq 0} \hat{V}^1(-k) \wedge \hat{V}^1(k) \left(i\frac{k}{|k|^2} \cdot B \right)$$

$$= \sum_{k_3=k_1=0} \hat{V}^1(-k) \wedge \hat{V}^1(k) \left(i\frac{k}{|k|^2} \cdot B \right) + \sum_{k_2=k_1=0} \hat{V}^1(-k) \wedge \hat{V}^1(k) \left(i\frac{k}{|k|^2} \cdot B \right)$$

$$= \frac{1}{2}\begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}.\left(\frac{i}{|j+1|^2}\begin{pmatrix} 0 \\ j+1 \\ 0 \end{pmatrix} \cdot B \right) + \frac{1}{2}\begin{pmatrix} 0 \\ -i \\ 0 \end{pmatrix}.\left(\frac{i}{|j+1|^2}\begin{pmatrix} 0 \\ -j-1 \\ 0 \end{pmatrix} \cdot B \right)$$

$$+ \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix}.\left(\frac{i}{|j+1|^2}\begin{pmatrix} 0 \\ 0 \\ j+1 \end{pmatrix} \cdot B \right) + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -i \end{pmatrix}.\left(\frac{i}{|j+1|^2}\begin{pmatrix} 0 \\ 0 \\ -j-1 \end{pmatrix} \cdot B \right) \quad (62)$$

$$= -\frac{1}{j+1}\begin{pmatrix} 0 \\ B_y \\ B_z \end{pmatrix}.$$

We do the same for the other ones:

$$\sum_{k\neq0} \hat{V}^2(-k)\wedge\hat{V}^2(k)\left(\boldsymbol{i}\frac{k}{|k|^2}\cdot B\right) = \frac{-1}{j+2}\begin{pmatrix}B_x\\0\\B_z\end{pmatrix}, \sum_{k\neq0}\hat{V}^3(-k)\wedge\hat{V}^3(k)\left(\boldsymbol{i}\frac{k}{|k|^2}\cdot B\right) = \frac{-1}{j+3}\begin{pmatrix}B_x\\B_y\\0\end{pmatrix}.$$

(63)

Finally, for all $i\in[\![1,3]\!]$, since the spectrum of U^j and V^i are disjoint:

$$\sum_{k\neq0}\hat{U}^j(-k)\wedge\hat{V}^i(k)\left(\boldsymbol{i}\frac{k}{|k|^2}\cdot B\right) = 0,$$

(64)

and then

$$\alpha^{(2)}(\tilde{U}) = \alpha^{(2)}(U^j) - \frac{\delta_1}{j+1}\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{\delta_2}{j+2}\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix} - \frac{\delta_3}{j+3}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}, \quad (65)$$

where the δ_i can be chosen as small as we want. Thus $\alpha^{(2)}(U)$ is as near as we want from a matrix of distinct non-zero eigenvalues, and it's the same for $\alpha^S(U)$ for $R_m < R_m^0$ (NB: the set in which R_m can be taken will vary continuously with U, thus the precaution we took at the beginning of the subsection to be sufficiently far from the limit)

2.5.3 End of the proof

In the sequel of this subsection, we now work with $\tilde{U}$ instead of U, droping the tilda. Thus, α^S has distinct non-zero eigenvalues. In Fourier transform, we now have to find the eigenvalues of the matrix $A^\xi\alpha^A + A^\xi\alpha^S$, where

$$A^\xi = \boldsymbol{i}\begin{pmatrix}0&-\xi_3&\xi_2\\\xi_3&0&-\xi_1\\-\xi_2&\xi_1&0\end{pmatrix}.$$

(66)

Let there be $P\in\mathscr{O}_3(\mathbb{R})$ such that

$${}^tP\alpha^S P = \begin{pmatrix}\alpha_1&0&0\\0&\alpha_2&0\\0&0&\alpha_3\end{pmatrix} = D,$$

(67)

and write $\zeta = {}^tP\xi$. The matrix $A^\zeta D$ has $\lambda_0 = 0$ eigenvalue, and two opposite eigenvalues given by

$$\lambda_\pm = \pm\sqrt{\zeta_1^2\alpha_2\alpha_3 + \zeta_2^2\alpha_3\alpha_1 + \zeta_3^2\alpha_1\alpha_2}.$$

(68)

Those eigenvalues are real for any ζ if all eigenvalues of α^S have the same sign. Otherwise, there exists a cone with vertex 0 outside of which they are. Furthermore, for ζ in the complement Ω of this cone, λ_+ stays positive, and λ_- stays negative. Thus, the eigenvectors

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of $A^\zeta D$ depend regularly of ζ. From now on, we fix some ζ in this good set Ω. We can also assume that $\xi = P\zeta \in (2\pi\mathbb{Q}_+^*)^3$.

Let β be an eigenvector associated to λ_+. Then $A^\zeta D\beta = \boldsymbol{i}\zeta \wedge (D\beta) = \lambda_+\beta$, so $\zeta \cdot \beta = 0$. Thus, we have $A^{\zeta\,t}P\alpha^A P\beta = \zeta \wedge (\gamma \wedge \beta) = -\boldsymbol{i}(\zeta \cdot \gamma)\beta$, where γ is a vector such that $\forall b \in \mathbb{R}^3, {}^tP\alpha^A Pb = \gamma \wedge b$. Thus β is also an eigenvector of $A^{\zeta\,t}P\alpha^A P$, associated with the eigenvalue $-\boldsymbol{i}(\zeta \cdot \gamma)$.

Now changing back to $\xi = P\zeta$, we have that

$$A^\zeta = {}^tPA^\xi P, \tag{69}$$

so that β is an eigenvector of ${}^tPA^\xi\alpha^A P + {}^tPA^\xi PD$ associated to the eigenvalue $\lambda_+ - \boldsymbol{i}(\zeta \cdot \gamma) = \lambda_+ - \boldsymbol{i}(\xi \cdot P\gamma)$. And thus $P\beta$ is an eigenvector of $A^\xi\alpha^A + A^\xi\alpha^S$ associated to an eigenvalue of positive real part, that is a growing mode solution of the first term of (38).

2.6   Conclusion to the linear case

We have proven now the existence of an eigenvector b_0^0 of the equation

$$P^0 A^1 P^0 B^\varepsilon + \varepsilon A'^\varepsilon B^\varepsilon = \lambda^\varepsilon B^\varepsilon \tag{70}$$

for $\varepsilon = 0$ with positive real part, that is an exponentially-growing mode of the form

$$B^\varepsilon(x, \theta, t) = b_0^0(\theta, \zeta)e^{(\lambda_+ - \boldsymbol{i}\zeta\cdot\gamma)t + \boldsymbol{i}\zeta\cdot x} \tag{71}$$

of the induction equation for given velocity field U_s and small enough R_m (and $\varepsilon = 0$). Since the operator $\frac{1}{\varepsilon}P^\varepsilon A^\varepsilon P^\varepsilon$ varies analytically on ε for small ε, and has distinct eigenvalues for $\varepsilon = 0$, so do eigenvalues and eigenvectors of this operator, and thus the result remains valid when ε is positive sufficiently small (at fixed ξ), and we have now an eigenvector b_0 of the induction equation for small enough ε.

This concludes the proof of theorem 2.

Remark 1 *Most of what we said stays valid if we take R_m out of a discrete set of values instead of only "small enough". Only the fact that $\alpha(U)$ is as near as we want to a matrix of distinct non-zero eigenvalues remains a problem in that case.*

Using the linear instability result in theorem 2, we will prove in the next section the instability of the full nonlinear MHD system (4).

3   Nonlinear instability

We will now prove nonlinear instability of the system (4). For that we will adapt a method developed by S. Friedlander, W. Strauss and M. Vishik in [5] (itself adapted from Y. Guo and W. Strauss in [12]) to prove nonlinear instability given linear instability. The condition $s > 3/2 + 1$ in the main theorem is used only in this section to obtain an inequality on

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the nonlinear terms similar to that in the article[5]. Actually, using the parabolic feature of the MHD system, it would be possible to lower our regularity requirements (see for instance S.Friedlander, N.Pavlović and R.Shvydkoy[4] in the context of the Navier-Stokes equations). We stick here to regular data for simplicity of exposure.

Let $U_s \in \mathscr{P}$ belonging to the dense subset of linearly unstable flows described in the previous section.

We introduce Leray operator $\mathbb{P}$ to get rid of p. We rewrite system (6) as

$$\partial_t \mathscr{U} + L_s \mathscr{U} = Q(\mathscr{U}, \mathscr{U}), \tag{72}$$

where

$$\mathscr{U} = \begin{pmatrix} u \\ b \end{pmatrix}, \tag{73}$$

$$L_s \mathscr{U} = \begin{pmatrix} \mathbb{P}(u \cdot \nabla U_s + U_s \cdot \nabla u) - \frac{1}{R_e} \Delta u \\ -\nabla \wedge (U_s \wedge b) - \frac{1}{R_m} \Delta b \end{pmatrix} \tag{74}$$

is the linearized operator around $\begin{pmatrix} U_s \\ 0 \end{pmatrix}$, and

$$Q(\mathscr{U}, \mathscr{U}) = \begin{pmatrix} \mathbb{P}((\nabla \wedge b) \wedge b - u \cdot \nabla u) \\ \nabla \wedge (u \wedge b) \end{pmatrix} \tag{75}$$

is the nonlinearity.

We shall consider this system in $\mathcal{H}_T^s$, T to be specified later. We also denote L_T^2 the L^2 space on the torus. Since we will work in these spaces until the end of the proof, we will drop the T in the notation. We have the following a priori estimates for the nonlinear term:

Every term of every component of $Q(\mathscr{U}, \mathscr{U})$ can be written as a product of one component of $\mathscr{U}$ and one of $\nabla \mathscr{U}$ (composed by $\mathbb{P}$). Thus, we have the first estimate

$$\|Q(\mathscr{U}, \mathscr{U})\|_{L^2} \leqslant C \|\mathscr{U} \cdot \nabla \mathscr{U}\|_{L^2} \leqslant C \|\mathscr{U}\|_{L^2} \|\nabla \mathscr{U}\|_{L^\infty}. \tag{76}$$

For $r = \frac{1}{2}(s + n/2 + 1) > n/2 + 1$ (here $n = 3$) and η such that $r = (1 - \eta)s$ (i.e. $\eta = \frac{1}{2} - \frac{n+2}{4s} = \frac{1}{2} - \frac{5}{4s}$), we have that

$$\|\nabla \mathscr{U}\|_{L^\infty} \leqslant \|\mathscr{U}\|_{\mathcal{H}^r} \leqslant C \|\mathscr{U}\|_{L^2}^\eta \|\mathscr{U}\|_{\mathcal{H}^s}^{1-\eta}. \tag{77}$$

Thus, we have:

$$\|Q(\mathscr{U}, \mathscr{U})\|_{L^2} \leqslant C \|\mathscr{U}\|_{L^2}^{1+\eta} \|\mathscr{U}\|_{\mathcal{H}^s}^{1-\eta} \tag{78}$$

We shall prove the following theorem, which is a mere reformulation of the first theorem:

Theorem 3 *Suppose $s > 3/2 + 1$, R_m small enough and $R_e > 0$. Then there exists a vector $T = (T_1, T_2, T_3)$ with integer coordinates such that the system (72) is nonlinearly unstable in $\mathcal{H}_T^s$ in the following sense:*

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There exists a growing mode $\mathcal{U}_0 = \begin{pmatrix} u_0 \\ b_0 \end{pmatrix}$ with $\|\mathcal{U}_0\|_{\mathcal{H}_T^s} = 1$ of the linearized system and a constant $C_0 > 0$ depending only on $\mathcal{U}_0, s, n, \rho$ such that for all $\delta > 0$, the perturbation of the linear solution $\delta \mathcal{U}_0$ satisfies $\|\mathcal{U}(t_\delta)\|_{\mathcal{H}_T^s} \geqslant C_0$ for some time t_δ. Furthermore, $t_\delta \sim_{\delta \to 0} -\frac{\ln \delta}{\rho}$.

Remark 2 *This theorem only proves that the solution reaches high values in finite time however small the initial condition is. In particular it doesn't say wether it explodes in finite time or is defined for all time, since both situations are possible in the result of the theorem. Actually, for small enough initial data (that is small enough δ), it is classical that the solution of the system doesn't explode in finite time and thus the approximation of t_δ for vanishing δ is valid. More details in [22] (Theorem 3.8)*

Assume in contrary that the system is nonlinearly stable while spectrally unstable, that is that for any growing mode of initial value $\mathcal{U}_0 = \begin{pmatrix} u_0 \\ b_0 \end{pmatrix}$ with $\|\mathcal{U}_0\|_{\mathcal{H}_T^s} = 1$ in any box T of the system (i.e. an eigenvector of L_s associated to an eigenvalue with positive real part) and for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $t > 0$, the solution $\mathcal{U}(t)$ of the system with initial values $\delta \mathcal{U}_0$ satisfies $\|\mathcal{U}(t)\|_{\mathcal{H}_T^s} \leqslant \varepsilon$.

3.1 Proof of the nonlinear instability

Let $b_L(\theta) = e^{i\varepsilon\xi\cdot\theta} b^\varepsilon(\theta)$ be the unstable eigenmode given by Theorem 2. We fix $\varepsilon < \varepsilon_0$ in such a way that

$$T = \left(\frac{2\pi}{\varepsilon|\xi_1|}, \frac{2\pi}{\varepsilon|\xi_2|}, \frac{2\pi}{\varepsilon|\xi_3|} \right) \tag{79}$$

belongs to $\mathbb{N}^3$. This is possible as $\xi \in (2\pi\mathbb{Q}_+^*)^3$ (take ε to be the inverse of a large integer). In particular, $\tilde{\mathcal{U}}_L = \begin{pmatrix} 0 \\ b_L \end{pmatrix}$ is an eigenvector of L_s associated to an eigenvalue with positive real part in $\mathcal{H}_T^s$.

Denote by ρ the maximum real part of the spectrum of L_s:

$$\rho = \max \left\{ \operatorname{Re} \lambda, \lambda \in \operatorname{Spectr}(L_s) \right\}. \tag{80}$$

By the previous remark on $\tilde{\mathcal{U}}_L$, we know that $\rho > 0$. Moreover, since the spectrum of L_s is only made of eigenvalues, there exists an eigenvector $\mathcal{U}_L = \begin{pmatrix} u_L \\ b_L \end{pmatrix}$ with eigenvalue λ satisfying exactly $\operatorname{Re} \lambda = \rho$.

Moreover, by standard properties of the spectral radius of e^{L_s}, we know that

$$e^\rho = \lim_{t \to +\infty} \|e^{t L_s}\|^{1/t}. \tag{81}$$

Thus, for all $\eta > 0$, there exists a C_η such that for all $t \geqslant 0$,

$$\|e^{t L_s}\| \leqslant C_\eta e^{\rho(1+\frac{\eta}{2})t} \tag{82}$$

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Then, the solution $\mathscr{U}$ of (72) with initial data δU_0 can be written

$$\mathscr{U} = \hat{\mathscr{U}} + \delta \mathscr{U}_L, \tag{83}$$

where $\hat{\mathscr{U}}$ satisfies

$$\begin{cases} \partial_t \hat{\mathscr{U}} = L_s \hat{\mathscr{U}} + Q(\mathscr{U}, \mathscr{U}) \\ \hat{\mathscr{U}}_{|t=0} = 0 \end{cases} \tag{84}$$

Using Duhamel's formula, we can write the solution under the form:

$$\hat{\mathscr{U}}(t) = \int_0^t e^{(t-s)L_s} Q(\mathscr{U}, \mathscr{U}) \, \mathrm{d}s. \tag{85}$$

Define

$$T_\delta = \sup \left\{ t > 0, \forall s \leqslant t, \|\hat{\mathscr{U}}(s)\|_{L^2} \leqslant \frac{1}{2} \delta e^{\rho s} \|\mathscr{U}_L\|_{L^2} \right\}, \tag{86}$$

where λ is the coefficient in the growing linear mode. Then, since we supposed that $\mathscr{U} = \mathscr{U}_l + \hat{\mathscr{U}}$ remains bounded, and $\|\mathscr{U}_l\| \sim \delta e^{\rho t} \|\mathscr{U}_L\|$, we have that $T_\delta < \infty$.

And for all $t < T_\delta$,

$$\|\mathscr{U}(t)\|_{L^2} \leqslant \frac{3}{2} \delta e^{\rho t} \|\mathscr{U}_L\|_{L^2}. \tag{87}$$

Using a priori estimates stated earlier, we have for $\eta = \frac{1}{2} - \frac{n+2}{4s} = \frac{1}{2} - \frac{5}{4s}$:

$$\|\hat{\mathscr{U}}(t)\|_{L^2} \leqslant C_\eta \int_0^t e^{\rho(1+\frac{\eta}{2})(t-s)} \|\mathscr{U}\|_{L^2}^{1+\eta} \|\mathscr{U}\|_{\mathcal{H}^s}^{1-\eta} \, \mathrm{d}s. \tag{88}$$

$$\begin{aligned} \|\hat{\mathscr{U}}(t)\|_{L^2} &\leqslant \delta^{1+\eta} C_\eta \int_0^t e^{(t-s)(1+\frac{\eta}{2})\rho} e^{(1+\eta)\rho s} \varepsilon^{1-\eta} \, \mathrm{d}s \\ &\leqslant C_\eta \delta^{1+\eta} e^{t(1+\frac{\eta}{2})\rho} \varepsilon^{1-\eta} \frac{2}{\eta \rho} (e^{t\frac{\eta}{2}\rho} - 1) \\ &\leqslant C' \delta^{1+\eta} e^{t(1+\eta)\rho} \varepsilon^{1-\eta} \end{aligned} \tag{89}$$

Thus, getting back to $\mathscr{U}$:

$$\|\mathscr{U}\|_{L^2} \geqslant \delta \|\mathscr{U}_L\|_{L^2} e^{\rho t} - C' \delta^{1+\eta} \varepsilon^{1-\eta} e^{t(1+\eta)\rho}. \tag{90}$$

By definition of T_δ, we have:

$$\|\hat{\mathscr{U}}(T_\delta)\|_{L^2} = \frac{1}{2} \delta e^{\rho T_\delta} \|\mathscr{U}_L\|_{L^2} \leqslant C' \delta^{1+\eta} \varepsilon^{1-\eta} e^{T_\delta(1+\eta)\rho}, \tag{91}$$

that is

$$\frac{1}{2} \|\mathscr{U}_L\|_{L^2} \leqslant C' \delta^\eta \varepsilon^{1-\eta} e^{T_\delta \eta \rho}, \tag{92}$$

$$T_\delta \geqslant \ln \left(\frac{\|\mathscr{U}_L\|_{L^2}}{2C' \delta^\eta \varepsilon^{1-\eta}} \right) \frac{1}{\eta \rho}. \tag{93}$$

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Denote by t_δ the quantity

$$t_\delta = \ln\left(\frac{\|\mathscr{U}_L\|_{L^2}}{2C'\delta^\eta\varepsilon^{1-\eta}(1+\eta)}\right)\frac{1}{\eta\rho}. \tag{94}$$

Thus $t_\delta \leqslant T_\delta$, and at this time t_δ

$$\|\mathscr{U}\|_{L^2} \geqslant \delta e^{\rho t_\delta}\left(\|\mathscr{U}_L\|_{L^2} - \frac{\|\mathscr{U}_L\|_{L^2}}{1+\eta}\right) = \frac{\eta}{1+\eta}\|\mathscr{U}_L\|_{L^2}\left(\frac{\|\mathscr{U}_L\|_{L^2}}{C'\varepsilon^{1-\eta}(1+\eta)}\right)^{1/\eta}. \tag{95}$$

Thus $\|\mathscr{U}\|_{L^2}$ (and $\|\mathscr{U}\|_{\mathcal{H}^s}$) can be bounded below by a value independent of δ, which contradicts the asumption, and proves the theorem.

3.2   Estimation of the growing time

We will now estimate the time in which $\|\mathscr{U}\|_{\mathcal{H}^s}$ becomes of order 1. Denote by T_0 this time:

$$T_0 = \sup\{t > 0, \forall s \leqslant t, \|\mathscr{U}(s)\|_{\mathcal{H}^s} \leqslant 1\}. \tag{96}$$

We use again the definition of T_δ:

$$T_\delta = \sup\{t > 0, \forall s \leqslant t, \|\hat{\mathscr{U}}(s)\|_{L^2} \leqslant \frac{1}{2}\delta e^{\rho s}\|\mathscr{U}_L\|_{L^2}\}. \tag{97}$$

This time, T_δ might be infinite. For big enough λ as in the former proof, we have again that

$$T_\delta \geqslant \ln\left(\frac{\|\mathscr{U}_L\|_{L^2}}{2C'\delta^\eta}\right)\frac{1}{\eta\rho} \tag{98}$$

Denote again by t_δ the right-hand side:

$$t_\delta = \ln\left(\frac{\|\mathscr{U}_L\|_{L^2}}{2C'\delta^\eta(1+\eta)}\right)\frac{1}{\eta\rho}. \tag{99}$$

Thus in time $t_0 = \inf\{t_\delta, T_0\}$ and with the constant

$$C_0 = \inf\left\{1, \frac{\eta}{1+\eta}\|\mathscr{U}_L\|_{L^2}\left(\frac{\|\mathscr{U}_L\|_{L^2}}{C'(1+\eta)}\right)^{1/\eta}\right\}, \tag{100}$$

we have

$$\|\mathscr{U}(t_0)\|_{L^2} \geqslant C_0, \tag{101}$$

and

$$t_0 \leqslant \ln\left(\frac{\|\mathscr{U}_L\|_{L^2}}{2C'(1+\eta)}\right)\frac{1}{\eta\rho} - \frac{\ln\delta}{\rho} \leqslant \frac{-\ln\delta}{\rho} + C_0'. \tag{102}$$

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